Homework Assignment 7

Dynamical Systems II

Bernold Fiedler, Hannes Stuke

http://dynamics.mi.fu-berlin.de/lectures/

due date: Thursday, December 04, 2014

Recall the definition of the shift space S_N

$$S_N := \Big\{ s = (s_j)_{j\in\mathbb{Z}} \mid s_j \in \{1,\dots,N\} \Big\}.$$

The topology of this space is generated by the cylinder sets $\mathcal{N}_k(s)$:

$$\mathcal{N}_k(s) := \{ \tilde{s} \in S_N \mid \tilde{s}_j = s_j \text{ for all } |j| \le k \}, \qquad s \in S_N, \quad k \in \mathbb{N}_0.$$

This means two things. First, a subset of S_N is open iff it is a (not necessarily finite or countable) union of sets $\mathcal{N}_k(s)$. Second, a sequence $s^{(n)}$ converges to s, in S_N, iff for every k there exists n_0 such that $s^{(n)} \in \mathcal{N}_k(s)$ for all $n \ge n_0$. For any $0 < \lambda < 1$ we also define a metric dist_λ on S_N:

$$\mathrm{dist}_\lambda(s,\tilde{s}) := \sum_{j\in\mathbb{Z}} \lambda^{|j|} \, |s_j - \tilde{s}_j| \, .$$

Problem 1: Consider the shift space S_N with the topology defined above.

(i) Prove that the cylinder sets $\mathcal{N}_k(s)$ are also closed. They are thus *open and closed*.

(ii) Use this fact to prove that S_N is totally disconnected, i.e. for arbitrary $s, \tilde{s} \in S_N$, $s \ne \tilde{s}$, there are open sets $U, \tilde{U} \subset S_N$, such that $s \in U$, $\tilde{s} \in \tilde{U}$, $U \cap \tilde{U} = \emptyset$, and $U \cup \tilde{U} = S_N$.

(iii) Prove, that S_N is (sequentially) compact, i.e. any sequence $s^{(n)}$ possesses a convergent subsequence.

Problem 2: Consider the shift

$$\sigma : S_N \to S_N, \quad \sigma(s)_j := s_{j-1}.$$

What are the orbits of σ with minimal period 2? Which s converge to these orbits under forward iteration of σ? Which s are homoclinic?

Problem 3: Prove or refute *one* of the following claims, for $0 < \lambda, \mu < 1$.

(i) The cylinder sets $\mathcal{N}_k(s)$ define the same open sets, i.e. the same topology, as any of the metrics dist_λ on S_N.

(ii) For any given λ, μ the metrics dist_λ, dist_μ are equivalent, i.e there exists $C \geq 1$ such that $C^{-1}\text{dist}_\lambda(s, \tilde{s}) \leq \text{dist}_\mu(s, \tilde{s}) \leq C\text{dist}_\lambda(s, \tilde{s})$ holds for all $s, \tilde{s} \in S_N$.

Extra credit: Prove or refute the other claim, too.

Problem 4: Consider arbitrary positive real parameters α, γ. Calculate all fixed points of the diffeomorphism f given by

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j + \gamma \sin(\Phi_j + v_j), \end{aligned}$$

with $\Phi_j \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $v_j \in \mathbb{R}$. How many fixed points does f possess, for given α, γ? Determine how the linearized type of the fixed points depends on the parameters: stable/unstable spiral/node, hyperbolic saddle, or non-hyperbolic. Sketch the regions of these types in the (α, γ)-plane. What happens for the area-preserving case $\alpha = 1$?